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Ray statistics beyond a fractal diffuser—an exactly soluble model

J H Jefferson

Royal Signals and Radar Establishment, St Andrews Road, Great Malvern, Worcs WR14 3PS, UK

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Abstract. A Brownian fractal random walk model for the gradient of rays scattered by random media is investigated. It is shown that when the emanating rays lie on a regular grid with slope given by the map $m_n = m_{n-1} + \varepsilon_n$, with $\varepsilon_n = \pm 1$, then their statistics and correlation properties may be determined exactly at all points where they intersect. The correlation length is shown to increase quadratically at large distances where the continuum limit (ray density) is discussed. The theory compares well with numerical simulations.

1. Introduction

Naturally occurring phenomena which show a random structure that appears similar over many length scales have been known and studied for many years, the most well known example being Brownian motion (see, e.g., Wax 1954). There has been a renewed interest in this subject in recent years following the work of Mandebrot (1982) who has emphasised the wide applicability of multiscale descriptions and has coined the generic word *fractal* to describe such objects which may be classified according to their *fractal dimension.*

An application area of considerable interest is that of waves which have encountered fractals, referred to as *diffractals* by Berry (1979) who also pointed out that no geometrical optics limit exists for such objects since their wavefronts are not differentiable. As a consequence of this, intensity fluctuations are very weak even though the scattering may be strong in the sense that the fractal scatterer may induce phase fluctuations which are many wavelengths. In contrast, wavefronts which are differentiable will give rise to focusing in the short-wave limit and hence very large intensity fluctuations, which may be analysed using catastrophe theory (see, e.g., Berry 1978). Jakeman (1982a) has argued that wavefronts which are once-differentiable, i.e. have gradients which are fractal, will possess a geometrical optics limit but, in the absence of higher-order derivatives, will not give rise to focusing. However, the intensity fluctuations can be large and take all values between those of a marginal diffractal (fractal slope dimension $D \to 2$) and a 'smooth' wavefront ($D \to 1$) (Jakeman and Jefferson 1984). Some discussion of scattered waves with these properties has been alluded to in the past, though without the benefit of a fractal description and interpretation (Rumsey 1975, Rino 1979).

Since wavefronts with fractal slopes give rise to intensity fluctuations covering a very wide range they represent an important classification of scatterers worthy of further study. In particular, we would like to determine the statistical and correlation properties of the scattered waves. Unfortunately this poses severe mathematical difficulties and even a determination of the second moment of intensity (contrast) has proved a laborious task. Indeed these difficulties are not confined to multiscale models and one usually has to resort to numerical methods or simulations. One exception is the case of the Brownian fractal-slope model ($D = 1.5$) which Jakeman (1982b) has solved exactly. It is the existence of this exact solution which motivated the present work. The intention was to simulate a Brownian fractal-slope wavefront on a computer and investigate the statistics and correlation properties of the emanating rays in the geometrical optics limit, comparing the results with the known analytic solution. We could then investigate the effects of finite size (inner and outer scale) and subsequently, with some confidence, apply the same computational techniques to other 'non-Brownian' fractal-slope wavefronts for which the statistics are not known.

When simulations were performed and the results compared with the analytic expressions the agreement was poor but improved with distance. These discrepancies led to a theoretical study of the 'discretised' model in its own right which was found to have some interesting mathematical properties and which also yielded exact solutions which compared very well with the simulations. The results of this ivestigation will be presented in the following sections. In § 2 the model is defined and in § 3 the ray statistics are determined at all points where the rays can intersect. Section 4 deals with the correlation properties which are shown to have an exponential decay in a certain limit and this enables the first-order ray-density statistics to be deduced in the continuum limit. These results are discussed in the final section where comparison with Monte Carlo simulations is made.

2. The model

Consider a plane wave which is passed through a narrow region of turbulence or reflected from a rough surface. If the scattering region is very narrow then, to a good approximation, only the phase of the wave will change as it traverses the scattering region. In the geometrical optics limit, the emanating waves may be described entirely by the random function $m(x, y)$, the gradient of the rays in the xy plane at $z = 0$. For mathematical simplicity, it will be assumed that the wavefronts are *corrugated*, i.e. $m(x, y)$ is constant for the lines $x = \text{constant}$. The wavefronts are thus determined by the scalar random function $m(x)$. It is further assumed that $m(x)$ is a *Brownian fractal* which satisfies the stochastic differential (Langevin) equation

$$\frac{\mathrm{d}m(x)}{\mathrm{d}x} = \frac{\varepsilon(x)}{l^{1/2}} \tag{1}$$

where $\varepsilon(x)$ is a δ-correlated random process, i.e.

$$\langle \varepsilon(x)\varepsilon(x')\rangle = \delta(x - x'). \tag{2}$$

With m and x replaced with velocity and time, equation (1) becomes the stochastic differential equation for ordinary Brownian motion in a 'frictionless' medium, the so-called Wiener process.

Some care is needed in the interpretation of (1) as a differential equation since, strictly speaking, $m(x)$ is not differentiable, though continuous. This point has received considerable attention in the literature and has been rigorously justified (see, e.g., Wax

1954). The formal solution to (1) is

$$m(x)=\frac{1}{l^{1/2}}\int_{-\infty}^{x}\varepsilon(x')\,\mathrm{d}x' \tag{3}$$

from which we easily deduce, using (2),

$$\langle m(x)^2\rangle=\infty \tag{4}$$

and

$$D(x)\equiv\langle[m(x+x')-m(x')]^2\rangle=|x|/l \tag{5}$$

where the angular brackets denote the ensemble average. Equation (5) merely states that the variance of increments in m is linear and independent of position, i.e. 'stationary', whilst equations (3) and (4) show that $m(x)$ is a Gaussian process with infinite variance irrespective of the statistics of $\varepsilon(x)$, provided they are stationary (by the central-limit theorem). It is also clear that $m(x)$ is a Markov process with

$$\begin{aligned}&P(m(x_1), m(x_2), m(x_3), \ldots)\\ &\quad=P(m(x_1))P(m(x_1)|m(x_2))P(m(x_2)|m(x_3))\ldots \qquad x_1<x_2<x_3\ldots\end{aligned} \tag{6}$$

where the conditional probabilities $P(m(x_i)|m(x_{i+1}))$ depend only on the increments $m(x_{i+1})-m(x_i)$.

To discretise the process $m(x)$ we generate a sequence of random numbers $m_0, m_1, m_2, \ldots$, on a uniform grid $x_0, x_1, x_2, \ldots$, where $x_n=n\Delta$. The m_n satisfy the recursion relation

$$m_n=m_{n-1}+(\Delta/l)^{1/2}\varepsilon_n \tag{7}$$

with

$$\langle\varepsilon_n\varepsilon_{n'}\rangle=\delta_{nn'} \tag{8}$$

giving

$$m_n=(\Delta/l)^{1/2}\sum_{i=0}^{n}\varepsilon_i \tag{9}$$

(cf equations (1)-(3)). It is easily verified that m_n satisfies equation (5) with $x=x_n=n\Delta$ and $x'=x_{n'}$.

As pointed out by Mandelbrot (1982), $m(x)$, like all fractals, satisfies an affine scaling law, i.e. remains statistically indistinguishable under the transformation $x\to kx$, $m\to k^{1/2}m$. This is shown in figure 1 where we successively shrink the z axis linearly and the x axis quadratically resulting in a RMS fluctuation in the ray gradient which is similar in all diagrams. It should be noted, however, that at certain distances (particularly at odd-integral z in figure 1(a)) the rays focus at points on a regular grid, though this 'fringing' disappears when viewed on a coarser scale (figure 1(d)). To some extent this focusing effect is a consequence of the fact that we have chosen the rays to originate from a uniform grid $\Delta=1$ with $\varepsilon_i=\pm1$. However, it is truly an inner-scale (discretisation) effect which is inescapable, for even if Δ were randomised and ε_i had some other statistics (but chosen so that equation (5) was still satisfied) there would still be 'fuzzy' focusing.

In the next section it will be shown that for the choice $\varepsilon_i=\pm1$ the focusing of rays at regular points may be exploited to yield exact solutions for the statistics and correlation properties of the converging rays. We conclude this section by remarking that the Gaussian-Markov process $m(x)$ may be defined rigorously as the limit $\Delta\to0$ of the well defined Markov chain defined by equations (7)-(9).

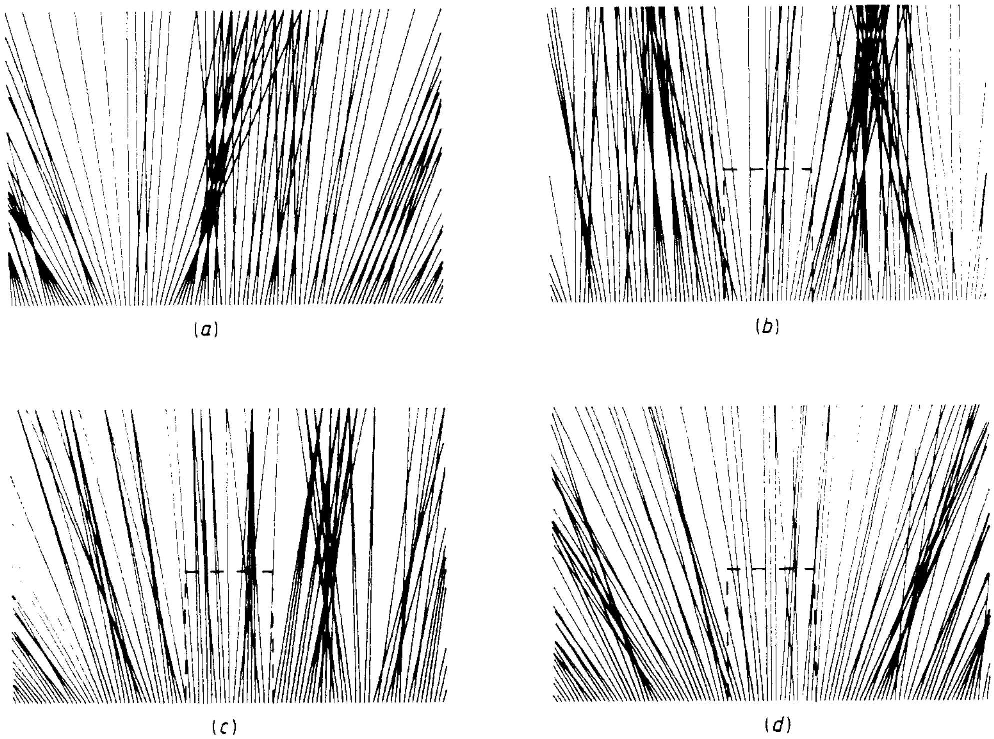

Figure 1. Ray diagrams under affine scaling transformations. Broken lines denote the boundary of the previous diagram.

3. Ray statistics

Choosing x in units of Δ and m in units of $(\Delta/l)^{1/2}$, equation (7) becomes

$$m_n = m_{n-1} + \varepsilon_n. \tag{10}$$

The equation of a ray from a point i at $z=0$ to a point x_i at z is

$$x_i = i + zm_i \tag{11}$$

and hence the separation of rays i and j is

$$x_j - x_i = j - i + z(m_j - m_i). \tag{12}$$

With the choice $\varepsilon_n = \pm 1$, m_n is an integer and hence for integer z, $x_j - x_i$ is also integral, i.e. all rays intersecting the line $z=$ integer lie on a regular lattice with unit spacing. Further, when z is an odd integer $x_j - x_i$ is either 2 or 0 (rays intersect) since $m_j - m_i$ is even when $j-i$ is even and odd when $j-i$ is odd. Hence, all rays intersecting the line $z=$ odd integer lie on a regular lattice with spacing 2. These regular focusing points may be seen clearly in figure 1(a).

Let n be the number of rays intersecting at a lattice point with z integral. The probability P_n that n rays will intersect may be determined indirectly by first deriving expressions for the factorial moments

$$\kappa_p(z) = \langle n(n-1)(n-2)\ldots(n-p+1)\rangle. \tag{13}$$

Since all lattice points are spatially equivalent in the statistical sense let us choose lattice points on the line $x = 0$ for convenience†. It also follows from spatial equivalence that

$$\langle n \rangle = 1$$

as is the case at $z = 0$ where $P_n = \delta_{n1}$, trivially.

In the appendix it is shown that

$$\kappa_p(z) = \sum_{i_1, i_2, \ldots, i_p = -\infty}^{\infty}{}' P\left(m_{i_1} = \frac{i_1}{z}, m_{i_2} = \frac{i_2}{z}, \ldots, m_{i_p} = \frac{i_p}{z}\right) \tag{14}$$

where $P(m_{i_1}, m_{i_2}, \ldots, m_{i_p})$ is the joint probability that the gradients at sites $i_1, i_2, \ldots, i_p$ will be $m_{i_1}, m_{i_2}, \ldots, m_{i_p}$ and the prime on the summation means that all the $i_1, \ldots, i_p$ must be different. Now P is symmetric in its arguments and hence

$$\kappa_p(z) = p! \sum_{i_1 < i_2 \ldots < i_p} P\left(m_{i_1} = \frac{i_1}{z}, m_{i_2} = \frac{i_2}{z}, \ldots, m_{i_p} = \frac{i_p}{z}\right). \tag{15}$$

Using the Markovian property (6) together with the normalisation condition

$$\sum_{i=-\infty}^{\infty} P(m_i = i/z) = 1$$

and the fact that the conditional probability

$$P(m_i = i/z \,|\, m_j = j/z)$$

depends only on $m_i - m_j = (i - j)/z$, equation (15) becomes, after replacing sums over $i_2, i_3, \ldots, i_p$ by sums over $i_2 - i_1, i_3 - i_2, \ldots, i_p - i_{p-1}$,

$$\kappa_p(z) = p!\left(\sum_{k=1}^{\infty} P(m_{j+k} - m_j = k/z)\right)^{p-1}. \tag{16}$$

Equation (9) tells us that $m_{j+k} - m_j = \varepsilon_{j+1} + \varepsilon_{j+2} + \ldots + \varepsilon_{j+k}$ must be integral since the $\varepsilon = \pm 1$ and hence k/z must be integral in (16). It follows that

$$P(m_{j+k} - m_j = k/z) = \delta_{k:lz} P(m_{j+k} - m_j = l) \tag{17}$$

with l integral. The probability that r of the ε are 1 and s are -1 is just the binomial factor ${}^{r+s}C_r/2^{r+s}$ and since

$$r + s = k = zl$$

$$r - s = k/z = l$$

then solving for r and s in terms of z and l and substituting in (17) gives

$$P(m_{j+k} - m_j = k/z) = \delta_{k:zl} \frac{{}^{zl}C_{l(1+z)/2}}{2^{zl}}. \tag{18}$$

Note that $r = l(1 + z)/2$ must be integral and hence l can take all positive integer values for z an odd integer but only even integers for z an even integer. Hence, substituting (18) into (17) and summing over k gives the exact result

$$\kappa_p(z) = p!\,\alpha^{p-1} \tag{19}$$

† When z is an odd integer only alternate lattice points of unit spacing are equivalent for one particular realisation. However, ensemble averaging ensures that site equivalence is regained.

where

$$\alpha(z)=\sum_{l=1}^{\infty}\frac{{}^{zl}C_{l(1+z)/2}}{2^{zl}} \tag{20a}$$

for z an odd integer and

$$\alpha(z)=\sum_{m=1}^{\infty}\frac{{}^{2zm}C_{m(1+z)}}{2^{2zm}} \tag{20b}$$

for z an even integer and $m=l/2$.

Inverting equation (19) (using the characteristic function $\langle(1+\theta)^n\rangle$) gives the probability distribution

$$P_n(z)=\frac{P_0(z)^{n+1}}{\alpha^2} \tag{21}$$

where $P_0(z)=\alpha/1+\alpha$.

The distribution is particularly simple at $z=1$ since

$$\alpha(1)=\sum_{l=1}^{\infty}2^{-l}=1$$

and hence

$$\kappa_p(1)=p! \tag{22}$$

giving

$$P_n(1)=(\tfrac{1}{2})^{n+1}. \tag{23}$$

This result can, in fact, be deduced directly as follows. Equation (12) tells us that the first focusing region is at $z=1$ and this is seen clearly in figure 1(a). Suppose a ray passes through a lattice point at $z=1$. The probability that an adjacent ray will also pass through this point is $\frac{1}{2}$ since it will either do so or intersect at the next lattice point with equal probability. Similarly, the probability that the $n\geqslant 1$ consecutive rays will pass through the same lattice point and the $(n+1)$th ray will not is $(\frac{1}{2})^{n+1}$. But this is the required probability, for once the chain of converging rays is broken no further rays can ever converge onto this same lattice point (see figure 1(a)). Thus the number of rays at different lattice sites are independent (simply because the increments of the Brownian walk are independent), i.e. they are δ correlated (see also the next section). This shows, in a very simple way, how the highly correlated Gauss–Markov process m_i is transformed into the δ-correlated process n_i with non-Gaussian statistics.

A further simplification arises at large distances since Stirling's formula

$$\frac{{}^{2mz}C_{m(1+z)}}{2^{2mz}}\xrightarrow[z\to\infty]{}(\pi mz)^{-1/2}\exp(-m/z) \tag{24}$$

may be used in (20) to give (replacing the sum by an integral)

$$\alpha(z=\text{odd integer})\xrightarrow[z\to\infty]{}2$$

and

$$\alpha(z=\text{even integer})\xrightarrow[z\to\infty]{}1$$

which are, in fact, upper bounds. Hence the factorial moments are

$$\begin{aligned}\kappa_p(z = \text{odd integer}) &\xrightarrow[z\to\infty]{} 2^{p-1}p! \\ \kappa_p(z = \text{even integer}) &\xrightarrow[z\to\infty]{} p!\end{aligned} \tag{25}$$

giving the probability distributions

$$P_{n>0} \to \tfrac{1}{4}(\tfrac{2}{3})^{n+1} \qquad \text{with } P_0 \to \tfrac{2}{3} \tag{26a}$$

for $z \to \infty$ an odd integer and

$$P_n \to (\tfrac{1}{2})^{n+1} \tag{26b}$$

for $z \to \infty$ an even integer. Note that (26*b*) is the same distribution as $P_n(1)$ though for quite different reasons.

The foregoing analysis will now be generalised to all points in the xz plane where rays can cross. From equation (12) it follows that since $j-i$ and $m_j - m_i$ must be integers then rays will cross $(x_j - x_i = 0)$ for values of z which are rational fractions $\geqslant 1$. Furthermore, given any real $z > 1$, we can always choose a rational fraction which is arbitrarily close to z. Let us consider a general point, $z = u/v$, where u and v are integers with $u \geqslant v$ (the case $u = v = 1$ has already been considered). From equation (12)

$$x_j - x_i = [(j-i)v + u(m_j - m_i)]/v \tag{27}$$

from which it follows that the rays at $z = u/v$ lie on a lattice with lattice parameter $2/v$ if u and v are both odd and $1/v$ otherwise. In particular, the lattice parameter is one when z is an even integer and two when z is an odd integer, as shown earlier.

The factorial moments and probabilities are again given by equations (19) and (21) with (cf equation (20))

$$\alpha(z = u/v) = \sum_{l=1}^{\infty} \frac{{}^{ul}C_{l(u+v)/2}}{2^{ul}} \tag{28a}$$

for $u + v$ even (i.e. u and v both odd) and

$$\alpha(z = u/v) = \sum_{m=1}^{\infty} \frac{{}^{2um}C_{m(u+v)}}{2^{2um}} \tag{28b}$$

for $u + v$ odd. When u is large Stirling's formula may again be applied through z need not now be necessarily large (e.g. $z = (v+1)/v$ with $v \to \infty$). Thus

$$\alpha(z = u/v) \xrightarrow[u\to\infty]{} \sum \left(\frac{2}{\pi u l}\right)^{1/2} \exp\left(\frac{-lv^2}{2u}\right) \to \frac{2}{v}\,\text{erfc}\left(\frac{v}{u^{1/2}}\right) \tag{29a}$$

for $u + v$ even and

$$\alpha(z = u/v) \xrightarrow[u\to\infty]{} \frac{1}{v}\,\text{erfc}\left(\frac{v}{u^{1/2}}\right) \tag{29b}$$

for $u + v$ odd.

Note that

$$\alpha \xrightarrow[v\to\infty]{} 0$$

and

$$P_{n>1} \xrightarrow[v\to\infty]{} 0$$

(see equation 21). This is clearly indicated in figure 1(a) where two or more rays never converge for z close to an integer (and hence v large). It is a consequence of the fact that converged rays can only originate from points separated by a distance u at $z=0$ (see equation (12)) and this is negligible for sufficiently large v.

Equations (19), (21) and (28) constitute an exact solution for the first-order statistics at $z \geq 1$.

4. Correlation function

Let us again, for convenience and clarity, restrict z to integral values. Denoting $n(x, z)$ as the (random) number of rays at point (x, z) then it follows from the appendix that the correlation function may be written

$$\begin{aligned} C(x, z) &= \langle n(x', z) n(x + x', z)\rangle \\ &= \sum_{kk'} P\left(m_k = \frac{k - x'}{z} : m_{k'} = \frac{k' - (x + x')}{z}\right) \\ &= \sum_{l=-\infty}^{\infty} P\left(m_{k+l} - m_k = \frac{l - x}{z}\right) \end{aligned} \tag{30}$$

where $l = k' - k$ and $\Sigma_k\, P(m_k = (k - x')/z) = 1$ has been used.

Note that not all values of l = integer will contribute to the sum in (30) since the corresponding probability is zero unless $m_{k+l} - m_k$ is an integer. Setting $(l - x)/z = p$, an integer, and summing over p gives

$$C(x, z) = \sum_{p=-\infty}^{\infty} P(m_{k+x+pz} - m_k = p) \tag{31}$$

ensuring that p and $l = x + pz$ are integral for x integral (otherwise $C(x, z) = 0$). However, not all values of negative p are allowed for certain x, e.g. when there is a solution $p = -x/z$.

The precluded p values may be determined as follows. Let r be the number of 1 and s the number of -1 in the random walk from $k \to k' = k + x + pz$. Then

$$r + s = |x + pz| \qquad r - s = p$$

that is,

$$r = \tfrac{1}{2}(|x + pz| + p) \qquad s = \tfrac{1}{2}(|x + pz| - p).$$

But $r \geq 0$ and $s \geq 0$, hence

$$|x + pz| \geq -p \tag{32a}$$

and

$$|x + pz| \geq p. \tag{32b}$$

For $p \geq 0$ both these equations are satisfied for all x and $z \geq 1$. For $p < 0$ (32b) is always satisfied but not (32a) which has precluded values in the range

$$-\frac{x}{z-1} < p < -\frac{x}{z+1}. \tag{33}$$

Hence, substituting the binomial probabilities ${}^{r+s}C_r/2^{r+s}$ into equation (31) gives finally

$$C(x,z) = \sum_{p=-\infty}^{\infty}{}' \frac{{}^{|x+pz|}C_{(|x+pz|+p)/2}}{2^{|x+pz|}} \tag{34}$$

where the prime on the summation means all p satisfying (33) are precluded†.

For the case $z=1$ it has already been argued in the last section that the number of rays at different sites are δ correlated. This may be demonstrated explicitly from equation (34) which becomes

$$C(x,1) = \sum_{p=-x/2}^{\infty} \frac{{}^{|x+p|}C_{(|x+p|+p)/2}}{2^{|x+p|}}$$

$$= \sum_{p=0}^{\infty} \frac{{}^{p+x/2}C_p}{2^{p+x/2}} = 2 \tag{35}$$

for x an even integer and 0 otherwise. Now $n(i,1)$, with i integral, is always zero for alternate sites since rays lie on a grid with spacing 2. Hence

$$\langle n(i,1)\rangle = \frac{0+2}{2} = 1$$

and

$$\langle n(i,1)\rangle\langle n(i+j,1)\rangle = \frac{0+4}{2} = 2$$

for j even and 0 otherwise, which is the same as $C(j,1)$ from equation (35). The rays at different sites are thus δ correlated.

The expression (34) can again be simplified when $z \gg 1$ when the precluded p values make a negligible contribution.

Applying Stirling's formula and replacing the sum by an integral gives

$$C(x,z) \xrightarrow[z\to\infty]{} \left(\frac{2}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} \frac{\exp(-p^2/2|x+pz|)}{(|x+pz|)^{1/2}}\, \mathrm{d}p$$

$$= 2[1+\exp(-2x/z^2)] \tag{36}$$

where x is an even integer and z an odd integer. In a similar way it is straightforward to show that

$$C(x,z) \xrightarrow[z\to\infty]{} 1+\exp(-2x/z^2) \tag{37}$$

when z is an even integer and x may be even or odd. Hence, for $x \ll z$, the rays are approximately fully correlated in the sense that (cf equations (25))

$$\langle n(x',z)n(x+x',z)\rangle \xrightarrow[\substack{z\to\infty \\ x/z^2\to 0}]{} \langle n(n-1)\rangle. \tag{38}$$

† For z an odd integer, p takes on all integer values apart from the precluded region (33). However, for z even, only even p are allowed when x is even and only odd p are allowed when x is odd (see equation (34)). For what follows z is taken to be odd though the results are easily extended to even z.

Using this fact it is straightforward to deduce the probability distribution for the ray density in the continuum limit as follows. The mean ray density over a distance k (integral) with z an even integer is

$$\bar{R}_k = \frac{1}{k} \sum_{i=1}^{k} n(i, z). \tag{39}$$

The pth moment is

$$\langle \bar{R}_k^p \rangle = k^{-p} \sum_{i_1, i_2, \ldots, i_p = 1}^{k} \langle n_{i_1} n_{i_2} \ldots n_{i_p} \rangle. \tag{40}$$

Now for $k \ll z^2$ the rays are approximately fully correlated and

$$\begin{aligned}\langle n_{i_1} n_{i_2} \ldots n_{i_p} \rangle &\simeq \langle n(n-1) \ldots (n-p+1) \rangle \\ &= p! \qquad (i_1 \neq i_2 \neq \ldots \neq i_p)\end{aligned}$$

from equation (25). Hence

$$\langle \bar{R}_k^p \rangle \underset{\substack{k, z \to \infty \\ k/z^2 \to 0}}{\longrightarrow} p!\, k^{-p} \sum_{i_1, i_2, \ldots, i_p = 1}^{k} 1 = p!$$

and in this limit

$$P(\bar{R}) = \exp(-\bar{R}) \tag{41}$$

a result obtained by Jakeman (1982b) directly for the continuum limit.

5. Summary and discussion

A discrete random walk model (equation (9)) for the gradient of rays scattered by random media has been solved exactly. The probability of n rays crossing at a point (equation (21)) was determined from expressions for the factorial moments (equation (19)). The results are particularly simple, and obvious, at $z = 1$ where the number of rays crossing at different 'sites' are uncorrelated. The results also simplify at large distances where Stirling's formula may be applied. This may be regarded as a consequence of the central limit theorem since $m_{j+k} - m_j$ will be approximately Gaussian when k is large, which it must be in equation (17) when z is large, i.e. rays crossing at a point at z can originate only from points separated by at least z. The correlation length was shown to increase from zero at $z = 1$ (δ correlated) to $\sim z^2$ for z large, where the decay of the correlation function is exponential. Thus, for $x \ll z^2$ rays become fully correlated and this fact was used to show that the ray density has an exponential probability distribution at large z.

The convergence to the continuum limit is quite slow as shown in figure 2 where $P_{1 \to 6}$ is plotted against z (an odd integer). To test the theory, Monte Carlo simulations with 100 000 rays were performed and results from these are also plotted in figure 2 with similar plots for the first six factorial moments in figure 3.

Although the agreement is good it should be pointed out that, in order to extract reasonably accurate asymptotic statistics from the simulations necessitated the generation of very large data sets. This is an unavoidable consequence of the multiscale nature of fractals and the requirement that many points need to contribute to the ray density in order that the effects of a finite step length (inner scale) become masked.

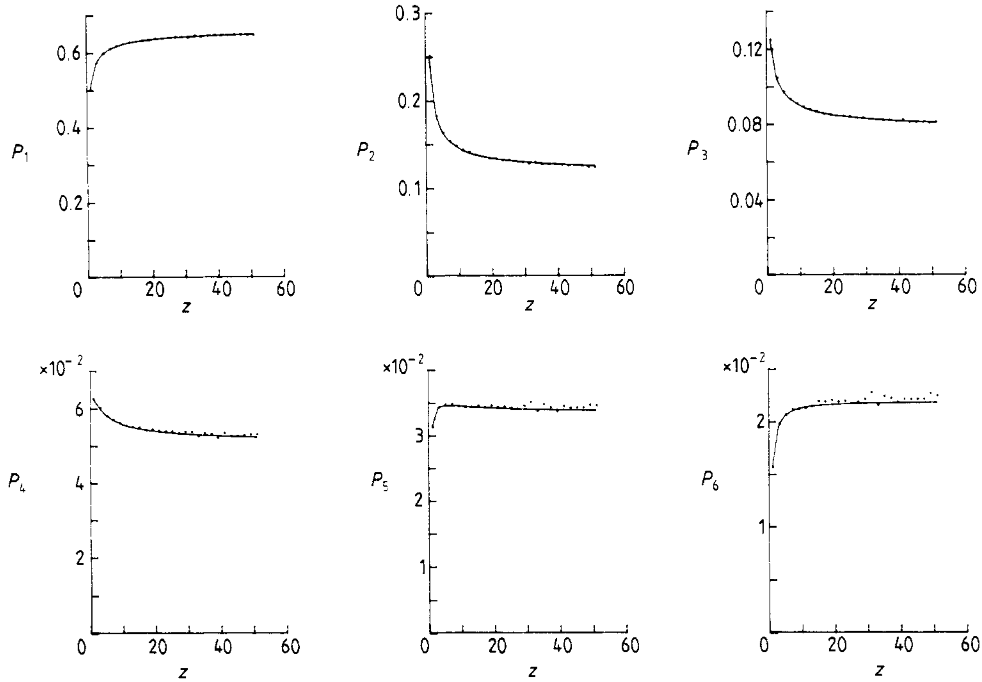

Figure 2. Probability distribution $P_{1\to 6}$ plotted against z. The full curve is the theoretical result and the points are Monte Carlo simulation results for 100 runs of 100 000 rays.

The results obtained here should be helpful for simulations of non-Brownian fractals where knowledge of the asymptotic ray density statistics is incomplete.

Acknowledgments

I am grateful for helpful and stimulating discussions with Drs E Jakeman and R J A Tough and for assistance with the computations from Mr J E P Beale and Miss J Edwards.

Appendix

Let $n \equiv n(x, z)$ be the number of rays at the point x, z, that is,

$$n = \sum_{i=-\infty}^{\infty} \delta_{m_i:(x_i-x)/z}.$$

Then

$$n(n-1) = \sum_i \delta_{m_i:(x_i-x)/z}\left(\sum_j \delta_{m_j:(x_j-x)/z} - 1\right)$$

$$= \sum_{\substack{i,j\\ i\neq j}} \delta_{m_i:(x_i-x)/z}\delta_{m_j:(x_j-x)/z} + \sum_i \delta_{m_i:(x_i-x)/z_i}(\delta_{m_i:(x_i-x)/z} - 1).$$

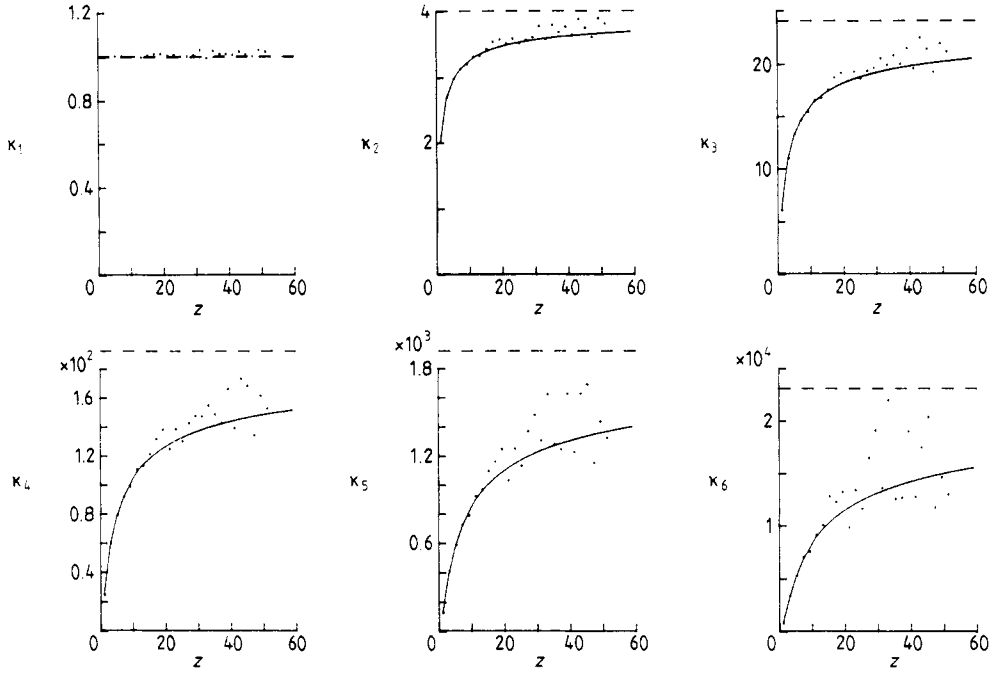

Figure 3. The first six factorial moments $\kappa_{1\to 6}$ plotted against z. The full curve is the theoretical result, the broken line is the asymptotic value and the points are Monte Carlo simulation results for 100 runs of 100 000 rays.

But $\delta_{m_i:(x_i-x)/z}(\delta_{m_i:(x_i-x)/z}-1)=0$ no matter what the value of m_i. Hence

$$n(n-1)=\sum_{\substack{i,j\\ i\neq j}} \delta_{m_i:(x_i-x)/z}\delta_{m_j:(x_j-x)/z}.$$

Similarly, it is straightforward to show that

$$n(n-1)(n-2)\ldots=\sum_{\substack{i,j,k\\ i\neq j\neq k}} \delta_{m_i:(x_i-x)/z}\delta_{m_j:(x_j-x)/z}\delta_{m_k:(x_k-x)/z}\ldots.$$

Averaging over the joint distribution $P(m_i, m_j, m_k, \ldots)$ gives

$$\langle n(n-1)(n-2)\ldots\rangle=\sum_{\substack{i,j,k\\ i\neq j\neq k}} P\left(m_i=\frac{x_i-x}{z}, m_j=\frac{x_j-x}{z}, \ldots\right).$$

In a similar way, all correlation functions may be expressed in terms of P.

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